INTEGER EXPONENTS AND DIVIDING MONOMIALS

Important Ideas

1. Exponents can be positive numbers, negative numbers, or zero.

2. All rules that apply to positive exponents also apply to negative exponents.

3. When a monomial has been simplified all of the exponents will be positive numbers.

4. To avoid dividing by zero we make the assumption that all variables represent positive or negative numbers, but not zero.

To Divide Monomials
1. Divide the numerical coefficients.
2. Divide like bases by subtracting exponents.

We will now work through some examples where all exponents are positive numbers and the larger exponent is in the numerator.

Example 1: Simplify: $\frac{x^5}{x^2}$

Before we apply the rule for dividing like bases, we will factor out the numerator and the denominator. This will help us to see why the rule works.

$$\frac{x^5}{x^2} = \frac{x\cdot x\cdot x\cdot x\cdot x}{x\cdot x}$$

Recall that when you are reducing fractions any factor that is in both the numerator and the denominator can be canceled, or divided out. This will leave us with three factors of "*x*" in the numerator.

$$\frac{\overset{1}{\cancel{x}}\cdot\overset{1}{\cancel{x}}\cdot x\cdot x\cdot x}{\underset{1}{\cancel{x}}\cdot\underset{1}{\cancel{x}}} = \frac{x^3}{1} = x^3$$

Instead of factoring we could apply the rule which says "**When dividing like bases, subtract the exponent in the denominator from the exponent in the numerator.**"

$$\frac{x^5}{x^3} = x^{5-3} = x^2$$

Example 2: Simplify: $\dfrac{45a^3b^7}{15ab^5}$

- Recall that if there is no exponent written it is understood to be "1".
- As 15 goes evenly into 45 we can divide the numerical coefficients.
- We will then subtract the exponents of like bases.

$$\frac{45a^3b^7}{15ab^5} = 3a^{3-1}b^{7-5} = 3a^2b^2$$

Example 3: Simplify: $\dfrac{-36x^4y^6}{8x^3y^2}$

Note that 8 does not divide evenly into −36. However, 8 and −36 do share a common factor of "4". We will divide both the numerator and the denominator by "4" and then subtract the exponents of like bases.

$$\frac{-\cancelto{9}{36}\,x^4y^6}{\cancelto{2}{8}\,x^3y^2} = \frac{-9x^{4-3}y^{6-2}}{2} = -\frac{9xy^4}{2}$$

Note that the final fraction was negative. This is because the numerical coefficients in the original fraction had different signs.

We will now look at some examples where the larger exponent is in the denominator.

Example 4: Simplify: $\dfrac{x^2}{x^6}$

As we did in Example 1 we will factor out the numerator and denominator first and cancel common factors.

$$\frac{x^2}{x^6} = \frac{\cancelto{1}{x}\cdot\cancelto{1}{x}}{\cancelto{1}{x}\cdot\cancelto{1}{x}\cdot x\cdot x\cdot x\cdot x} = \frac{1}{x^4}$$

Now let's see what happens when we apply the rule for dividing like bases and subtract the exponents. Recall that we will be subtracting the exponents in the denominator from the exponent in the numerator.

$$\frac{x^2}{x^6} = x^{2-6} = x^{-4}$$

Often when we subtract exponents we will get a negative exponent. It is okay to get a negative exponent but it's not okay to leave it in that form. **Any base that has a negative exponent must be rewritten in an equivalent form with a positive exponent.**

$$x^{-4} = \frac{1}{x^4}$$

You can see from the two different ways that we simplified this problem that the two solutions must be equivalent as they are both solutions to the <u>same</u> problem.

Any base (except zero) that has a negative exponent can be rewritten in an equivalent form by writing a fraction where the numerator is "1" and where the denominator is the base with a positive exponent.

For example: $y^{-6} = \frac{1}{y^6}$ and $a^{-8} = \frac{1}{a^8}$

Let's try another example.

Example 5: Simplify: $\frac{xy^3}{x^2y^5}$

Subtract the exponents. $\frac{xy^3}{x^2y^5} = x^{1-2}y^{3-5} = x^{-1}y^{-2}$

Rewrite negative exponents. $= \frac{1}{x} \cdot \frac{1}{y^2}$

Write as a single term. $= \frac{1}{xy^2}$

Example 6: Simplify: $\frac{30x^4y^7z^2}{27x^5y^4z^6}$

Note that 27 will not divide evenly into 30. However, they do share a common factor of "3".

We will divide both the numerator and the denominator by 3 and then subtract the exponents of like bases.

$$\frac{30x^4 y^7 z^2}{27x^5 y^4 z^6} = \frac{10x^{4-5} y^{7-4} z^{2-6}}{9} = \frac{10x^{-1} y^3 z^{-4}}{9}$$

Rewrite negative exponents. $= \frac{10}{9} \cdot \frac{1}{x} \cdot \frac{y^3}{1} \cdot \frac{1}{z^4} = \frac{10y^3}{9xz^4}$

In the previous examples the exponents in the original monomial were all positive. It is quite possible that the exponents in either the numerator or the denominator could be negative.

Example 7: Simplify: $\frac{x^{-2} y^{-4}}{x^{-4} y^3}$

We will divide like bases by subtracting the exponents.

Be very careful with the signs. $\frac{x^{-2} y^{-4}}{x^{-4} y^3} = x^{-2-(-4)} y^{-4-3}$

$= x^{-2+4} y^{-4+(-3)}$

$= x^2 y^{-7}$

Rewrite with a positive exponent. $= x^2 \cdot \frac{1}{y^7}$

Write as a single term. $= \frac{x^2}{y^7}$

Example 8: Simplify: $\frac{2^{-1} a^3 b^{-4}}{2^2 a^{-4} b^{-5}}$

Divide like bases by subtracting the exponents.

$$\frac{2^{-1}a^3b^{-4}}{2^2a^{-4}b^{-5}} = 2^{-1-2}a^{3-(-4)}b^{-4-(-5)}$$

Be careful with the signs. $= 2^{-1+(-2)}a^{3+4}b^{-4+5}$

$= 2^{-3}a^7b^1$

Rewrite with a positive exponent $= \dfrac{1}{2^3}\cdot a^7b^1$

Write as a single term and simplify the denominator. $= \dfrac{a^7b}{8}$

We have now had several examples where the exponents were negative or positive numbers. Let's have a look now at what happens if the exponent is a zero.

Example 9: Simplify $\dfrac{x^3}{x^3}$

As we did in examples 1 and 4 we will first factor out the numerator and the denominator and cancel common factors.

$$\frac{x^3}{x^3} = \frac{\overset{1}{\cancel{x}}\cdot\overset{1}{\cancel{x}}\cdot\overset{1}{\cancel{x}}}{\underset{1}{\cancel{x}}\cdot\underset{1}{\cancel{x}}\cdot\underset{1}{\cancel{x}}} = \frac{1}{1} = 1$$

Note that there were an equal number of factors in both the numerator and the denominator, and that after we canceled common factors we were left with only "1" in both the numerator and the denominator.

This is really a number divided by itself which is always equal to the number "1".

Let's see what happens if we apply the rule for dividing like bases and subtract the exponents.

$$\frac{x^3}{x^3} = x^{3-3} = x^0$$

You can see from the two different ways that we simplified this problem that the two solutions must be equivalent as they are both solutions to the <u>same</u> problem. <u>In fact any number (or expression), except zero, raised to the zero power is always equal to "1".</u>

$(xy)^0 = 1$ $\quad$ $(5a\,b^2)^0 = 1$ $\quad$ $\left(53xy^{24}\right)^0 = 1$ $\quad$ $5\,1^0 =$

Recall that one of the important ideas on page 1 was that all the rules that apply to positive exponents also apply to negative exponents. A second important idea was that when a monomial has been simplified all exponents will be positive.

The previous examples have all involved division of monomials. We will now review multiplying like bases and simplifying powers of exponential expressions.

<u>Rule for Multiplying Like Bases</u>
When multiplying like bases (same variables or number base), keep the base and <u>add</u> the exponents.

Example 10: Simplify: $x^{-4} \cdot x^{-5}$

Keep the base and add the exponents. $\quad x^{-4+(-5)} = x^{-9}$

Rewrite with a positive exponent. $\quad = \frac{1}{x^9}$

Example 11: Simplify: $4^{-6} \cdot 4^2$

Keep the base and add the exponents. $\quad 4^{-6+2}$

$= 4^{-4}$

Rewrite with a positive exponent. $\quad = \frac{1}{4^4}$

Simplify the denominator. $\quad = \frac{1}{256}$

Note that as the base was a known number rather than a variable that the denominator could be simplified.

Rule for Simplifying Powers of Exponential Expressions

To simplify powers of exponential expressions, multiply the exponents.

Example 12: Simplify: $\left(x^3\right)^{-4}$

Multiply the exponents. $x^{3(-4)} = x^{-12}$

Rewrite with a positive exponent. $= \dfrac{1}{x^{12}}$

Example 13: Simplify: $\left(3^{-2} x y^{4} {}^{-3}\right)^2$

Multiply the exponent of each base inside the parentheses by the exponent outside the parentheses. $3^{(-2)(2)} x^{(4)(2)} y^{(-3)(2)}$

$= 3^{-4} x y^{8} {}^{-6}$

Rewrite with positive exponents. $= \dfrac{1}{3^4} \cdot \dfrac{x^8}{1} \cdot \dfrac{1}{y^6}$

Simplify and write as a single term $= \dfrac{x^8}{81y^6}$

There are a few problems in the textbook where the negative exponent is in the denominator rather than the numerator. We will discuss the rule that applies in this case.

If n is a positive integer and $x \neq 0$, then $\dfrac{1}{x^{-n}} = x^n$

You can see that the base has moved from the denominator to the numerator and that the exponent is now positive. Once you understand how this occurs you can move the base and change the sign of the exponent.

$\dfrac{1}{x^{-n}} = \dfrac{1}{\left(\dfrac{1}{x^n}\right)}$ Rewrite the denominator with a positive exponent.

$$= 1 \div \frac{1}{x^n}$$

Recall that a fraction bar is a way of saying division.

$$= 1 \times \frac{x^n}{1}$$

To divide fractions, multiply by the reciprocal of the divisor (denominator)

$$= x^n$$

You can see that the base is now in the numerator and the exponent is positive.

EXAMPLE 14: Simplify: $\frac{x^4}{y^{-3}}$

$$\frac{x^4}{y^{-3}} = \frac{x^4 y^3}{1} = x^4 y^3$$

Move the base from the denominator into the numerator and change the sign of the exponent.

<u>Practice Exercises</u>

1. $x^5 y^{-3}$

3. $\frac{3^{-2}}{3^{-1}}$

5. $(a^2 b^4)^0$

7. $\frac{6a^2}{18a^6}$

9. $\frac{-28x^{-2}y^3}{16x^3y^{-5}}$

<u>Answers to Practice Exercises:</u>

1. $\frac{x^5}{y^3}$

2.

3. $\frac{1}{3}$

4.

2. $\frac{a^{-5}}{a^{7}}$

4. $\left(x\, y^{-4}{}^{2}\right)^{-2}$ 5. 1 6. 7. $\frac{1}{3a^4}$ 8. $-\frac{15}{x^4}$

6. $\left(-3x\, y^{-2}{}^{3}\right)^{2}$ 9. $\frac{7y^8}{4x}$ $-$ $\frac{}{5}$ 10.

8. $\frac{1}{x^{-5}}$ $\frac{1}{a^{12}}$

10. $\left(-3x^{-6}\right)\left(5x^{2}\right)$ x^8

$\frac{}{y^4}$

$\frac{9y^6}{x^4}$

x^5