Factoring Trinomials in the Form of $ax^2 + bx + c$ by Grouping

The method of factoring by grouping can be used as an alternative to factoring trinomials by trial and error. Let us recall the procedure for multiplying binomials using FOIL:

Simplify: $(x+2)(x+3)$

$x^2 + \; + \; +3x \; 2x \; 6$

$x^2 + \; +5x \; 6$

Now reverse directions and factor the trinomial:

$x^2 + 5x + 6$ Find factors of 6 that add to 5. They are 2 and 3. $(x+2)(x+3)$

When we factor trinomials of the form $x^2 + bx + c$, we look for the factors of c (the constant) that add to b (the coefficient of the first degree term). Actually, what we are doing is looking for the factors of 1(the coefficient of the squared term) times c (the constant) that add to b (the coefficient of the first degree term). Thus, when we want to factor trinomials of the form $x^2 + bx + c$, we look for factors of ac (the product of the coefficient of the squared term and the constant) that add to b (the coefficient of the first degree term).

Now let us see how factoring by grouping can also work:

$x^2 + 5x + 6$ Find factors of 6 (which is the product of 1 and 6 where 1 is the coefficient of the squared term and 6 is the constant) that add to 5 (which is the coefficient of the first degree term) and rewrite $5x$ as a sum of two first degree terms whose coefficients are

$x^2 + 2x + 3x + 6$ pair off the terms

$(x^2 + 2^x) + (3x+6) \; x$ Factor out the GCF of each pair.

Notice that this is not the answer yet. We have terms, not factors.

$x(\; + \; +2) \; 3(x+2)$ Now factor out the common binomial factor.

This is the factored form.

$(x+2)(x+3)$

Example

those factors

Multiply using FOIL: $(2x+3 \; 3)(\; x+4)$

$6x^2 + \; + \; +8x \; 9x \; 12$

$$6x^2+17x+12$$

Now reverse directions and factor the above:

$6x^2+17x+12$	Find 2 factors of 72 (the product of 6 and 12) which add to 17. They are 9 and 8. Replace $17x$ with $9x + 8x$
$6x^2+9x+8x+12$	Notice that this matches the 2nd line in the multiplication of binomials above. Now group the terms in pairs.
$(6x^2+9x)+(8x+12)$	Factor out the GCF from each pair of terms.
$3x(2x+3)+4(2x+3)$	Factor out the common binomial factor.
$(2x+3)(3x+4)$	This is the factored form. Notice that this matches the original multiplication problem above.

When we changed $17x$ to $9x + 8x$ in the example above, we also could have used $8x + 9x$. Since addition is commutative, we should get the same result. We begin again:

$$6x^2+17x+12$$
$$6x^2+8x+9x+12$$
$$(6x^2+8x)+(9x+12)$$
$$2x(3x+4)+3(3x+4)$$
$$(3x+4)(2x+3)$$

Notice that we have the same two factors but in the reversed order. Since multiplication is commutative, this does not matter and both answers are equally correct.

Example Factor $2x^2 - x - 21$ by grouping.

$2x^2-x-21$	Find the factors of -42 $(2\times(-21))$ which add to -1. That would be -7 and 6. Rewrite $-x$ as $-7x + 6x$
$2x^2-7x+6x-21$	Group the terms in pairs.
$(2x^2-7x)+(6x-21)$	Factor out the GCF from each pair.
$x(2x-7)+3(2x-7)$	Factor out the common binomial factor.
$(2x-7)(x+3)$	This is the factored form.

If we had reversed the order of the $-7x$ and $6x$ on the second line, we would have the following work:

$2x^2 + 6x - 7x - 21$

Since the third term is negative, remember to change the signs in the second pair of parentheses when you factor out -1.

$\left(2x^2 + 6x\right) - \left(7\ 21x +\right)$

Factor out the GCF of each pair.

$2x\ x\left(\ + -3\right)\ 7\left(x+3\right)$

Factor out the common binomial factor.

$\left(x+3\ 2\right)\left(\ x-7\right)$

Notice that it was a little easier the first time, when we made the second term negative and the third one positive.

Example Factor: $3x^2 + 13x - 10$

$3x^2 + 13x - 10$

$3\ (\ 10)\cdot\ -= -30$

$= 3x^2 -\ +2x\ 15x - 10$

Factors of -30 that add to 13 are 15 and -2

$= \left(3x^2 - 2^x\right) + \left(15x - 10\right)$

Notice that we purposely put the negative term second and the positive term third.

$= x\left(3x-\ +2\right)\ 5\ 3\left(\ x-2\right)$

$= \left(3x-2\right)\left(x+5\right)$

Example Factor: $2x^2 - 7x + 6$

$2x^2 -\ +7x\ 6$

$2x^2 -\ -\ +3x\quad 4x$

$6\ \left(2x^2\right.$

$2\ 6\ 12\cdot\ =$

$\left.-3^x\right) - \left(4x-6\right)$

Factors of 12 that add to -7 are -3 and -4.

$x\left(2x-\ -3\right)\ 2\ 2\left(\right.$

$\left.x-3\right)$

Since both the second and third terms are negative, we will have to factor out a negative from the latter pair of terms.

$(2x-3)(x-2)$

When attempting to factor any trinomial, remember to always look for a common factor first.

Example

Factor $6x^3 + 22x^2 + 20x$

$2\ 3x\ x(^2 + 11\ 10x + \quad)$

$2\ 3x\ x[\lfloor^2 + 6\ 5\ 10x + x + \quad\rfloor]$

$2\ 3x[\lfloor(\ x^2 + 6x) + (5\ 10x +)\rfloor]$

$2\ 3x\ x\ x[\lfloor(\ + 2\ 5) + (x + 2)\rfloor]$

$2x\ x(\ + 2\ 3\ 5)(\ x +)$

The GCF is $2x$.

$310\ 30\cdot \ =$

Find factors of 30 that add to 11. (6 and 5)

Group the terms into pairs. Change parentheses to brackets in preparation for the next step where we will have nested grouping symbols.

Factor out the GCF of each pair of terms.

Factor out the common binomial factor from the two terms. This is the factored form.

Example Factor: $30x^5 + 5x^4 - 75x^3$

$5\ 6x^3(\ x^2 + 1\ 15x - \quad)$

$5x^3\ [\lfloor 6x^2 - 9x + 10x - {}^{15}\rfloor]\ 5x^3$

$\lceil(6x^2 - 9^x) + (10x - 15^)\rceil]$

$\lfloor$

$5x^3\ [\lfloor 3x(2x-3) + 5\ 2(\ x-3)\rfloor]$

$5x^3(2x-3)(3x+5)$

Factors of -90 that add to 1 are -9 and 10

Example Factor: $27x^3 - 39x^2 + 12x$

Factors of 36 that add to -13 are -9 and -4

$3x(9x^2 - 13x + 4)$

$3x(9x^2 - 9x - 4x + 4)$

$3x[(9x^2 - 9x) - (4x - 4)]$

$3x[9x(x - 1) - 4(x - 1)]$

$3x(x - 1)(9x - 4)$

Not every trinomial will factor. For example, try to factor $5x^2 + 7x + 9$.

$5x^2 + 7x + 9$

First we check for a common factor other than 1. There is none. Then we look for factors of 45 that add to 7. The factor pairs for 45 are:

1, 45	1 + 45 = 46
3, 15	3 + 15 = 18
5, 9	5 + 9 = 14

None of these add to 7.

Therefore $5x^2 + 7x + 9$ is a *prime* polynomial—its only factors are itself and one, and it cannot be factored into two binomials. We say that this polynomial is non-factorable over the integers.

Example Factor $3x^2 + 7x - 12$

$3x^2 + 7x - 12$ There is no common factor other than 1.

$$3 \cdot (-12) = -36$$

Factors of -36 are:

$1, -36$	$1 - 36 = -35$	$-1, 36$	$-1 + 36 = 35$
$2, -18$	$2 - 18 = -16$	$-2, 18$	$-2 + 18 = 16$
$3, -12$	$3 - 12 = -9$	$-3, 12$	$-3 + 12 = 9$
$4, -9$	$4 - 9 = -5$	$-4, 9$	$-4 + 9 = 5$
$6, -6$	$6 - 6 = 0$		

Since none of these pairs add to 7, there will be no binomial factors. Therefore this polynomial is prime and is non-factorable over the integers.

EXERCISES

1. $2x^2 + x - 15$
2. $3x^2 - 11x + 10$
3. $3x^2 + 13x + 4$
4. $7x^2 + 8x + 1$
5. $3x^2 + 4x - 8$
6. $6x^2 + x - 1$

9. $21x^2 - 13x + 2$
10. $20x^2 + 39xy - 11y^2$
11. $8x^2 + 47x - 6$
12. $16x^2 + 30xy + 9y^2$
13. $24x^2 - 37x - 5$
14. $15x^2 - 39x + 18$

7. $12x^2 + 11x - 5$

15. $18x^4 + 15x^3 - 75x^2$

16. $12x^{3\ 4}y - 4x\ y^{2\ 5} - xy^1$

8. $20x^2 - 28xy - 3y^2$

Answers

1.	$(2x-5)(x+3)$	7.	$(4x+5\ 3)(\ x-1)$	12.	$(8x+3y)(2x+3y)$
2.	$(3x-5)(x-2)$	8.	$(10x+y)(2x-3y)$	13.	$(3x-5\ 8)(\ x+1)$
3.	$(3x+1)(x+4)$	9.	$(7x-2\ 3)(\ x-1)$	14.	$3\ 5(\ x-3)(x-2)$
4.	$(7x+1)(x+1)$	10.	$(5x+11y)(4x\ y-\ )$	15.	$3x^2(3x-5\ 2)(\ x+5)$
5.	prime	11.	$(8x-1)(x+6)$	16.	$xy^4(6x\ y+\ )(2x\ y-\ )$

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1 . $(3x-1\ 2)(\ x+1)$

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