Factoring by Grouping

Factors are expressions joined by multiplication. In the expression $3x$, the factors are 3 and x. In the expression $4(x + 2)$, the factors are 4 and $(x + 2)$. The factor $(x + 2)$ is called a binomial factor since it has two terms. In the expression $3x(x + 4) + 5(x + 4)$, $3x(x + 4)$ and $5(x + 4)$ are terms because they are joined by addition. The factor $x + 4$ is called the <u>common binomial factor</u> because it is common to both terms.

When an expression has a common factor, we can write it as a product of factors. For example, if we are asked to factor $3x^2 + 6x$, we factor out the greatest common factor which is $3x$. Then we divide each of the original terms by that GCF to get the other factor:

$$\frac{3x^2}{3x} + \frac{6x}{3x} = x + 2$$

The factored form of $3x^2 + 6x$ is $3x(x + 2)$. In this case we were factoring out the common <u>monomial</u> factor.

In the expression $3x(x + 4) + 5(x + 4)$, we can also factor out the greatest common factor. This time the GCF is the binomial $(x + 4)$. If we factor out $(x + 4)$ in the same manner that we factor out monomial factors, we would bring the common factor, $(x + 4)$, out front and divide each term of the original expression by that binomial to get the other factor.

$$\frac{3x(x + 4)}{x + 4} + \frac{5(x + 4)}{x + 4} = 3x + 5$$

The factored form of $3x(x + 4) + 5(x + 4)$ is $(x + 4)(3x + 5)$.

We can check to see if we are correct using the following steps:

1. Be sure that there is no common factor, other than the number 1, in either pair of parentheses.

2. Simplify both the original expression and your answer and check to see that they are the same.

Checking the above problem:

1. $x + 4$ has no common factor other than the number 1.

   $(3x + 5)$ has no common factor other than the number 1.

2. Compare the two forms by simplifying:

   Simplify $3x(x + 4) + 5(x + 4)$ Distribute, then collect like terms

   $3x^2 + 12x + 5x + 20$

   $3x^2 + 17x + 20$

   Simplify $(x + 4)(3x + 5)$ Use FOIL

   $3x^2 + 5x + 12x + 20$

   $3x^2 + 17x + 20$

Notice that these are the same.

Example: Factor $7a(x + 2) - 3(x + 2)$

$(x + 2)$ is the GCF

Divide each term by $(x + 2)$ and get $7a - 3$

The factored form is $(x + 2)(7a - 3)$

<u>Factoring when a binomial involves differences</u>

We know that 5 and -5 are opposites of each other. The opposite of 5 is written $-(5)$ and is -5. Likewise, the opposite of $(x - 4)$ is $-(x - 4) = -1(x - 4) = -x + 4 = 4 + -(x) = 4 - x$.

If we reverse the order in a subtraction problem, the result we get is the opposite of the expression in the original problem. For example, $7 - 3 = 4$, but $3 - 7 = -4$, which is the opposite of 4. Therefore, $7 - 3$ and $3 - 7$ are opposites. In general, $a - b$ and $b - a$ are opposites of each other. In the example above, $x - 4$ and $4 - x$ are opposites.

Example: Find the opposite of $x - 6$

Answer: $6 - x$

When we want to factor a problem such as $5x(x - 2) + 3(2 - x)$, we need to notice that $x - 2$ and $2 - x$ are opposites of each other. Before we can see the common factor, we must rewrite one of the two binomials. We usually rewrite the one that is not in descending order. For this example, that means that we should rewrite $2 - x$ as $-(x - 2)$ or $-1(x - 2)$.

$5\,x\,(x-2)+3\,2(-x)$

$5^{x}\,(x-{}^{2})+{}^{3}\left[\lfloor-{}^{1}(x-{}^{2})\rfloor\right]$ Now use the associative property for multiplication

on the second term. 5^{x}

$(x-{}^{2})+\left[\lfloor{}^{3}(-{}^{1})\rfloor\right](x-{}^{2})$ Simplify the bracket.

$5^{x}\,(x-{}^{2})-{}^{3}(x-{}^{2})$ Now we can factor out the GCF which is $x-2$. $(x-$

$2)(5x-3)$

As a shortcut, when we take the opposite of a binomial such as $2-x$, we can write the opposite of the sign on that term because that is what multiplying by -1 does. We can now write the work for the above problem as follows:

$5\,x\,(x-2)+3\,2(-$

$x)\;5x\,(x-2)-3(x-$

$2)$

$(x-2)(5x-3)$

Example: Factor: $2y\,(y-3)+5\,3(-y)$ Since $y-3$ and $3-y$ are opposites, as we change the $3-y$

to $y\;-3$, we must change the sign on that term. $2y\,(y-3)-5(y-3)$

$(y-3)(2y-5)$

Example: Factor: $4^{a}\,(x-{}^{7})-{}^{9\,7}(-{}^{x})$

$4a\,(x-7)+9(x-7)$

$(x-7)(4a+9)$

Example: Factor: ${}^{x}\,(2y-{}^{1})+{}^{5\,1}(-2^{y})\;x$

$(2y-1)-5\,2(y$

$-1)$

$(2y-1)(x-5)$

<u>Factoring Expressions with Four Terms</u>

In some variable expressions which have four terms, we can factor by grouping. For example:

$x^3-2x^2+5x-10$ We can first pair off the terms.

$(x^3-2x^2)+(5x-10)$ Then factor out the GCF of each pair.

$x^2(x-2)+5(x-2)$ This is not yet in factored form because it is a sum of two

terms. Now we see that $x-2$ is the GCF of the two terms in this expression. Factor it out.

$(x-2)(x^2+5)$ This is in factored form because it is a <u>product</u>.

Notice that the goal is to get a common binomial factor, such as $x-2$, so that it is possible to do the last step.

Example: Factor: $ab-3a+bc-3c$ Pair off the terms.

$(ab-3a)+(bc-3c)$ Factor out the GCF of each pair.

$a(b-3)+c(b-3)$ Factor out the GCF of these two terms.

$(b-3)(a+c)$

Example: Factor: $5x^3+10x^2+4x+8$

$(5x^3+10x^2)+(4x+8)$

$5x^2(x+2)+4(x+2)$

$(x+2)(5x^2+4)$

If the third term is negative, it will be necessary to factor out a negative GCF from the third and fourth terms in order to get a common binomial factor. For example, if we want to factor $x^3-3x^2-5x+15$, we begin by grouping the first two terms

$$(x^3-3x^2)-5x+15$$

Realize that because of the minus sign in front of the $5x$, we cannot place the parentheses around $5x+15$ because $-5x+15$ is not the same as $-(5x+15)$. The latter is equal to $-5x-15$. Therefore, when we have a negative sign on the third term of the original expression, we must factor out a common -1 from the last two terms. Notice that $-5x+15$ is the same as $-1\ 5(x\ -15)$ or $-(5x-15)$.

$x^3-3x^2-5x+15$ becomes

$(x^3-3x^2)-(5x-15)$ Now we can continue, by factoring the GCF out of each pair of terms.

$x^2(x-3)-5(x-3)$ Factor out the GCF of these two terms.

$(x-3)(x^2-5)$

Example: Factor: $4ax - 8x - 3a + 6$ Be sure to change the signs on both $-3a$ and $+6$ when grouping the last two terms. $(4ax - 8x) - (3a - 6)$

$4x(a - 2) - 3(a - 2)$

$(a - 2)(4x - 3)$

If we had factored out $+3$ from the last two terms instead, we would have had

$4ax - 8x - 3a + 6$

$(4ax - 8x) + -(3a + 6)$

$4x(a - 2) + 3(-a + 2)$

Notice that $a - 2$ and $-a + 2$ are opposites. Now we would have to take the opposite of $-a + 2$ and change the sign on that term.

$4x(a - 2) - 3(a - 2)$

$(a - 2)(4x - 3)$

As you can see, this is the same answer, but it made the work longer.

Example: Factor $x^3 - 3x^2 - 6x + 18$

$(x^3 - 3x^2) - (6x - 18$

$) x^2(x - 3) - 6(x -$

$3)$

$(x - 3)(x^2 - 6)$

Sometimes a common factor of a pair of terms is 1 or -1 as illustrated by these two examples:

Example: Factor $x^3 - 4x^2 + x - 4$

$(x^3 - 4x^2) + (x - 4)$

$x^2(x - 4) + 1(x - 4)$ Be sure to write that 1 in front of the $(x - 4)$

$(x - 4)(x^2 + 1)$

Example: Factor $ax + a - x - 1$ Factor out -1 from the last pair.

$a(x + 1) - 1(x + 1)$ Be careful of those ones! Remember to change the signs on the last two terms. $(x$

$+1)(a-1)$

Exercises:

1. $a\,x\,(+\,y\,)+7\,(x+y\,)$
2. $4x\,(3y-8\,)-5\,3(\,y-8)$
3. $3x^2\,(a-9\,)+6\,9(\,-a)$
4. $2m\,p\,(-1)-9\,1(\,-p\,)$
5. $3\,(x-4\,)+5x\,(4-x)$
6. $x^2\,(x+2\,)+(x+2)$
7. $(a-3)+5b(a-3)$
8. $5b(2a-1)-(2a-1)$
9. $ab+5a+bc+5c$
10. $a^2+2a+ab+2b$
11. $x^2-2x+xy-2y$
12. $cx-cy-dx+dy$
13. $cx-cy-dy+dx$
14. $5c+15-2cd-6d$
15. $4x+24+ax+6a$
16. x^3-3x^2+2x-6
17. $18x^2+12xy-3xy-2y^2$
18. $3a^3+3ab^2+2a\,b^2+2b^3$
19. $2ab+6b-5a-15$
20. $2x^2y+8x+xy+4$
21. $3ab+b^2-3a-b$
22. a^3-3a^2+2a-6
23. $3x^3+6x^2+2x+4$
24. $ax-ay-bx+by$
25. $2abcd-2cd-ab+1$

Answers:

1. $(x+y)(a+7)$
2. $(3y-8)(4x-5)$
3. $(a-9)(3x^2-6)$
4. $(\,p-1)(2m+9)$
5. $(x-4)(3-5x)$
6. $(x+2)(x^2+1)$
7. $(a-3)(1+5b)$
8. $(2a-1)(5b-1)$
9. $(b+5)(a+c)$
10. $(a+{}^{2})(a+{}^{b})$
11. $(x-{}^{2})(x+y)$
12. $(x-y)(c-d\,)$
13. $(x-y)(c+d\,)$
14. $c\,3\,5-2\,d$
15. $(x+6)(4+a)$
16. $(x-3)(x^2+2)$
17. $(3x+2y)(6x-y)$
18. $(a^2+b^2)(3a+2b)$
19. $(a+3)(2b-5)$
20. $(xy+4)(2x+1)$
21. $(3a+b)(b-1)$
22. $(a-3)(a^2+2)$
23. $(x+2)(3x^2+2)$
24. $(x-y)(a-b)$
25. $(ab-1)(2cd-1)$