Logarithms

A logarithm of a given number x, is the exponent required for the base a, to be raised to in order to produce that number x.

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

Note that $\Leftrightarrow$ means "*is equivalent to*"

Logarithmic and Exponential Form

Change logarithm equations to exponential form or exponential equations to logarithmic form using the definition of a logarithm.

Example: Given $4^{3/2} = 8$, change the equation to logarithmic form.

Solution:

Compare the equation to the definition and rewrite it.

Definition: $\log_a x = y \Leftrightarrow a^y = x$ ⟺ Notice that $a = 4$, $x = 8$, and $y = \frac{3}{2}$, respectively.

Given: $4^{3/2} = 8$

Therefore, using the definition: $4^{3/2} = 8 \Leftrightarrow \mathbf{\log_4 8 = \frac{3}{2}}$

Example: Given $\log_{25} 5 = \frac{1}{2}$, change the equation to exponential form.

Solution:

Compare the equation to the definition and rewrite it.

Definition: $\log_a x = y \Leftrightarrow a^y = x$ ⟺ Notice that $a = 25$, $x = 5$, and

Given: $\log_{25} 5 = \frac{1}{2}$ $y = \frac{1}{2}$, respectively.

Therefore, using the definition: $\log_{25} 5 = \frac{1}{2} \Leftrightarrow \mathbf{25}^{1/2} = \mathbf{5}$

Solving Logarithm and Exponential Equations

Evaluate logarithmic equations by using the definition of a logarithm to change the equation into a form that can then be solved.

Example: Given $3^{x-1} = 7$, solve for x.

Solution:

Step 1: Set up the equation and use the definition to change it.

Definition: $\log_a x = y \Leftrightarrow a^y = x$

Given $3^{x-1} = 7$

Notice 3 is the base or a, and 7 is the given number.

$\Leftrightarrow \log_3 7 = x - 1$

Step 2: Now use the properties of logarithms to solve.

Recall the Change of Base Property:

$\log_a b = \frac{\log b}{\log a}$

Apply it to $\log_3 7$.

$\log_3 7 = \frac{\log 7}{\log 3}$

$3^{\frac{\log 7}{\log 3}} = 7$

Step 3: Use the order of operations to finish solving for x.

$x - 1 = \frac{\log 7}{\log 3}$

$\boldsymbol{x = \frac{\log 7}{\log 3} + 1}$

Example: Given $\log_6(x + 2) = 3$, solve for x.

Solution:

Step 1: Set up the equation and use the definition to change it.

Definition: $\log_a x = y \Leftrightarrow a^y = x$
Given $\log_6(x + 2) = 3$
Notice 6 is the base or a, and
3 is the exponent or y.

$\log_6(x + 2) = 3 \Leftrightarrow 6^3 = x + 2$

Step 2: Now use the order of operations to solve.

$6^3 = x + 2$
$216 = x + 2$
$214 = x$

$\boldsymbol{x = 214}$

Expanding and Simplifying Logarithms

To expand or simplify logarithms, utilize the various properties of logarithms in conjunction with the definition.

Example: Given $\log_3\left(\frac{9x^2}{\sqrt{x^2+1}}\right)$, expand the logarithm.

Solution:

Step 1: Expand the expression using the properties of logarithms.

Recall the Logarithm Multiplication and Division Properties:
$\log_a mn = \log_a m + \log_a n$

m
$\log_a\left(—\right) = \log_a m - \log_{n a} n$

Apply them to $9x^2$ and $\sqrt{x^2+1}$.

Given $\log_3\left(\frac{9x^2}{\sqrt{x^2+1}}\right)$:

$\Rightarrow \log_3 9 + \log_3 x^2 - \log_3\left(\sqrt{x^2+1}\right)$

so our final answer becomes:

Step 2: Now simplify further using the properties of logarithms and the definition.

Recall the Logarithm for Powers Property:
$\log_a x^c = c\log_a x$
Apply it to the x^2 and $\log_3\left(\sqrt{x^2+1}\right)$.

$\log_3 9 + \log_3 x^2 - \log_3\left(\sqrt{x^2+1}\right)$

$\Rightarrow \log_3 9 + \log_3 x^2 - \log_3(x^2+1)^{1/2}$

$\Rightarrow \log_3 9 + 2\log_3 x - \frac{1}{2}\log_3(x^2+1)$

By definition, $\log_3 9 = 2$ since $3^2 = 9$,

$\mathbf{2 + 2\log_3 x - 1} \quad - \quad {}^{\mathbf{2}}\mathbf{+ 1)}$
$\mathbf{\log_3(x\ 2}$

Example: Write $3\log_2 y - \log_2 x - 7\log_2 z$ as a single logarithm.

Solution:

To simplify the expression, work backwards with the logarithmic properties.

Step 1: Use the Logarithm for Powers Property where appropriate.

Given: $3\log_2 y - \log_2 x - 7\log_2 z$
Notice that it can be applied to $3\log_2 y$ and $7\log_2 z$.

$3\log_2 y - \log_2 x - 7\log_2 z$
$\Rightarrow \log_2 y^3 - \log_2 x - \log_2 z^7$

Step 2: Simplify using the Logarithm Multiplication and Division Properties.

Use the order of operations as a guide.

$\log_2 y^3 - \log_2 x - \log_2 z^7$
$\Rightarrow \log_2 y^3 - (\log_2 x + \log_2 z^7)$
$\Rightarrow \log_2 y^3 - \log_2 xz^7$
$\Rightarrow \log_2 \frac{y^3}{xz^7}$

Solving Expanded Logarithms

Solving expanded logarithms requires applying the definition of logarithms and all the logarithm properties as needed.

Example: Given $\ln(x-2) + \ln(x-3) = \ln(2x+24)$, solve for x.

Solution:

Note: $\ln(x-2)$ is only valid if $x \geq 2$, $\ln(x-3)$ is only valid if $x \geq 3$, and $\ln(2x+24)$ is only valid if $x \geq -12$. For the equation to be valid, all conditions must be met, so $x \geq 3$.

Step 1: Simplify the left side of the equation using the multiplication and division properties of logarithms.

$\ln(x-2) + \ln(x-3) = \ln(2x+24)$

$\Rightarrow \ln(x-2)(x-3) = \ln(2x+24)$
$\Rightarrow \ln(x^2 - 5x + 6) = \ln(2x+24)$

Step 2: Use logarithm properties. Recall logarithm properties of bases:
$\ln e^x = x$ and $e^{\ln x} = x$

$\ln(x^2 - 5x + 6) = \ln(2x+24)$

Let both sides of the equation become the exponent of the base e, and apply the property.

$\Rightarrow e^{\ln(x^2-5x+6)} = e^{\ln(2x+24)}$

$\Rightarrow x^2 - 5x + 6 = 2x + 24$

Practice Exercises:

1. Given $\log_4(-x) + \log_4(6-x) = 2$, Solve for x.

2. Expand $\log_2\left(\frac{x}{\sqrt{x^2-1}}\right)$ completely.

3. Write the following as a single logarithm: $2\log_3 x + 4 - 8\log_3 y$ Step 3: Combine like terms to solve for x.

$x^2 - 5x + 6 = 2x + 24$

$\Rightarrow x^2 - 7x - 18 = 0$

$\Rightarrow (x-9)(x+2) = 0$

$x = 9, -2$

Step 4: Check your answers. Recall that every logarithm must meet the conditions for the answer to be correct.

For $x = 9$

$\ln((9)-2) + \ln((9)-3) = \ln(2(9)+24)$

$\Rightarrow \ln(7) + \ln(6) = \ln(42)$

$\Rightarrow \ln(7 \cdot 6) = \ln(42)$ ⇢ This is valid!

For $x = -2$

Since $-2 \ngeq 3$, it does not meet all the conditions, and is not valid.

Therefore: $\boldsymbol{x = 9}$

Answers:

1. $x = -2$
2. $\log_2 x - \frac{1}{2}\log_2(x-1) - \frac{1}{2}\log_2(x+1)$
3. $\log_3 \frac{81x^3}{y^8}$