Logarithms

A logarithm of a given number *x*, is the exponent required for the base *a*, to be raised to in order to produce that number *x*.

 $\log_a x = y \quad \Leftrightarrow \quad a^y = x$

Note that \Leftrightarrow means "is equivalent to"

Logarithmic and Exponential Form

Change logarithm equations to exponential form or exponential equations to logarithmic form using the definition of a logarithm.

Example: Given 4

 $3^{2} = 8$, change the equation to logarithmic form.

Solution:

Compare the equation to the definition and rewrite it.

Definition: $\log_a x = y \Leftrightarrow a^y = x$ 3/2 = 8Given: 4
Notice that a = 4, x = 8, and 3 $y = \overline{2}$, respectively.

Therefore, using the definition: $4^{3/2} = 8 \Leftrightarrow \log_4 8 = 3_2$

Example: Given $\log_{25} 5 = \frac{1}{2}$, change the equation to exponential form.

Solution:

Compare the equation to the definition and rewrite it.

Definition: $\log_a x = y \Leftrightarrow a^y = x$

Notice that a = 25, x = 5, and

Given: $\log_{25} 5 = \frac{1}{2}$ $y = \frac{1}{2}$,

 $\frac{1}{2} \qquad 1/2 = 5$ Therefore, using the definition: $\log_{25} 5 = \iff 25$

Solving Logarithm and Exponential Equations

Evaluate logarithmic equations by using the definition of a logarithm to change the equation into a form that can then be solved.

Example: Given $3^{x-1} = 7$, solve for *x*.

Solution:

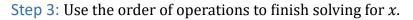
Step 1: Set up the equation and use the definition to change it.

Definition: $\log_a x = y \Leftrightarrow a^y = x$ Given $3^{x-1} = 7$

Notice 3 is the base or *a*, and 7 is the given number.

 $\Leftrightarrow^{x-1} \log_3 7 = x - 1$

$$\log_3 7 =$$



 $x - 1 = \frac{\log 7}{\log 3}$ $x = \frac{\log 7}{\log 7} + 1$ $\log 3$

Example: Given $\log_6(x + 2) = 3$, solve for *x*.

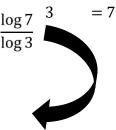
Solution:

logarithms to solve.

Step 2: Now use the properties of

Recall the Change of Base Property: $\log b$ $\log_a b = \underline{a}$ \log

Apply it to log₃ 7.



 $y = \frac{1}{2}$, respectively.

| Step 1: Set up the equation and use the definition to change it. | Step 2: Now use the order of operations to solve. |
|--|---|
| Definition: $\log_a x = y \Leftrightarrow a^y = x$ Given $\log_6(x + 2) = 3$ Notice 6 is the base or <i>a</i> , and 3 is the exponent or <i>y</i> . | $6^3 = x + 2$ 216 = x + 2 214 = x |
| $\log_6(x+2) = 3 \Leftrightarrow 6^3 = x+2$ | x = 214 |

Expanding and Simplifying Logarithms

To expand or simplify logarithms, utilize the various properties of logarithms in conjunction with the definition.

Example: Given $\log_3\left(\frac{9x^2}{\sqrt{x^2+1}}\right)$, expand the logarithm. Solution:

Step 1: Expand the expression using the properties of logarithms.

Step 2: Now simplify further using the properties of logarithms and the definition.

Recall the Logarithm Multiplication and Division Properties: $\log_a mn = \log_a m + \log_a n$

$$m \log_a \left(-\right) = \log_a m - \log_{n_a n}$$

Apply them to $9x^2$ and $\sqrt{x^2 + 1}$.

 $\log_3\left(\frac{9x^2}{\sqrt{x^2+1}}\right)_{:}$

 $\Rightarrow \log_3 9 + \log_3 x^2 - \log_3 \left(\sqrt{x^2 + 1} \right)$ so our final answer becomes: Recall the Logarithm for Powers Property: $\log_a x^c = c\log_a x$ Apply it to the x^2 and $\log_3(\sqrt{x^2 + 1})$

$$\log_{3} 9 + \log_{3} x^{2} - \log_{3} \left(\sqrt{x^{2} + 1} \right)$$

$$\Rightarrow \log_{3} 9 + \log_{3} x^{2} - \log_{3} (x^{2} + 1)^{1/2}$$

$$\Rightarrow \log_{3} 9 + 2 \log_{3} x - \frac{1}{2} \log_{3} (x^{2} + 1)$$

By definition, $\log_3 9 = 2$ since $3^2 = 9$,

 $2 + 2 \log_3 x - 1 - \frac{2}{\log_3(x \ 2)} + 1)$

Example: Write $3 \log_2 y - \log_2 x - 7 \log_2 z$ as a single logarithm.

Solution:

To simplify the expression, work backwards with the logarithmic properties.

Step 1: Use the Logarithm for Powers Property where appropriate.

Given: $3 \log_2 y - \log_2 x - 7 \log_2 z$ Notice that it can be applied to $3 \log_2 y$ and $7 \log_2 z$.

 $3 \log_2 y - \log_2 x - 7 \log_2 z$ $\Rightarrow \log_2 y^3 - \log_2 x - \log_2 z^7$ Step 2: Simplify using the Logarithm Multiplication and Division Properties. Use the order of operations as a guide.

 $\log_2 y^3 - \log_2 x - \log_2 z^7$ $\Rightarrow \log_2 y^3 - (\log_2 x + \log_2 z^7)$ $\Rightarrow \log_2 y^3 - \log_2 xz^7$ $\Rightarrow \frac{y_3}{xz} \log_2 z^7$

Solving Expanded Logarithms

Solving expanded logarithms requires applying the definition of logarithms and all the logarithm properties as needed.

Example: Given $\ln(x - 2) + \ln(x - 3) = \ln(2x + 24)$, solve for *x*.

Solution:

Note: $\ln(x - 2)$ is only valid if $x \ge 2$, $\ln(x - 3)$ is only valid if $x \ge 3$, and $\ln(2x + 24)$ is only valid if $x \ge -12$. For the equation to be valid, all conditions must be met, so $x \ge 3$.

Step 1: Simplify the left side of the equation using the multiplication and division properties of logarithms.

 $\ln(x-2) + \ln(x-3) = \ln(2x+24)$

 $\Rightarrow \ln(x-2)(x-3) = \ln(2x+24)$ $\Rightarrow \ln(x^2 - 5x + 6) = \ln(2x+24)$

Step 2: Use logarithm properties. Recall logarithm properties of bases: $\ln e^x = x$ and $e^{\ln x} = x$

 $\ln(x^2 - 5x + 6) = \ln(2x + 24)$

Let both sides of the equation become the exponent of the base *e*, and apply the property.

 $\Rightarrow e \ln(x^2 - 5x + 6) = e \ln(2x + 24)$

 $\Rightarrow x^2 - 5x + 6 = 2x + 24$

Practice Exercises:

1. Given $\log_4(-x) + \log_4(6-x) = 2$, Solve for x.

2. Expand $\log_2\left(\frac{x}{\sqrt{x^2-1}}\right)$ completely.

3. Write the following as a single logarithm: $2 \log_3 x + 4 - 8 \log_3 y$ Step 3: Combine like terms to solve for *x*. $x^2 - 5x + 6 = 2x + 24$ $\Rightarrow x^2 - 7x - 18 = 0$ $\Rightarrow (x - 9)(x + 2) = 0$ x = 9, -2

Step 4: Check your answers. Recall that every logarithm must meet the conditions for the answer to be correct.

For x = 9 $\ln((9) - 2) + \ln((9) - 3) = \ln(2(9) + 24)$ $\Rightarrow \ln(7) + \ln(6) = \ln(42)$ $\Rightarrow \ln(7 \cdot 6) = \ln(42) \longrightarrow$ This is valid! For x = -2Since $-2 \ge 3$, it does not meet all the conditions, and is not valid.

Therefore: x = 9

Answers:

- 1. x = -22. $\log_2 x - \frac{1}{2} \log_2(x-1) - \frac{1}{2} \log_2(x+1)$ 81 x^3
- 3. log₃ _____y₈